Competitive Equilibrium with Complete Markets: Part II

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ECON8022, This version April 8, 2008
ARC EDN Winter School on Money & Pricing

**When:** 04 Jul 2008 - 08 Jul 2008  
**Where:** Melbourne University

“Money and Pricing” will consist of three main components: lectures in monetary theory (Randy Wright, UPenn) and pricing (Preston McAfee, CalTech/Yahoo!), and presentations by graduate students.

Participants may also be interested in attending the *3rd Annual Workshop on Macroeconomic Dynamics*, which takes place at The University of Melbourne directly preceding the Winterschool: July 2-3. This workshop includes Professors Randall Wright (University of Pennsylvania) and Steven Turnovsky (University of Washington) as plenary speakers.

Outline

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2. Pareto (recap)
3. ADE ⇔ PO
4. SME
   - A state variable
   - SBC and Debt limits
   - Timing
   - Agents’ problems
5. ADE ⇔ SME
6. RCE
   - Recursive formulation
   - Markovian Asset pricing
   - Arbitrage-free pricing and redundant assets
What Next?

- Competitive/Decentralized equilibrium:
  - Arrow-Debreu time-0-trading economy (ADE) with Arrow-Debreu securities.
  - Radner sequential-trading economy (SME) with Arrow securities.

- Show ADE $\Leftrightarrow$ SME $\Leftrightarrow$ PO.
Motivation

Previously ...

Previously we look at a *planned* economy:

- Single optimizing planner.
- Characterized recursive optimal allocations as a DP problem.
- In reality, we have *decentralized* or *competitive economies.*

Osamu Tezuka’s *Metropolis*
And few lectures ahead ... 

- Want to work toward the stochastic growth model as also basic recursive competitive equilibrium model.
- As a dynamic outcome where individuals and firms solve their decentralized optimal allocation problems independently.
- No planner.
- History (or State)-contingent, intertemporal relative prices, as the allocative mechanism.
- Resulting versions of first- and second fundamental welfare theorems. (Why?)
Model Setup

- Stochastic event $s_t \in S = \{s_1, ..., s_n\}$ for $t \in \mathbb{N}$.
- Publicly observable history of events up to and including $t$: $h^t = (s_0, s_1, ..., s_t) \in S^t$.
- Unconditional probability of $h^t$ given by probability measure $\pi_t(h^t)$.
- W.l.o.g., assume $\pi(s_0) = 1$.
- Probability of observing $h^t$ conditional on realization of $h^\tau$ is $\pi(h^t|h^\tau)$, for any $t \geq \tau$. 
Model Setup

- \( I \) agents indexed by \( i = 1, \ldots, I \).
- Agent \( i \)'s
  - Endowment: \( y^i_t (h^t) \)
  - history-dependent consumption plan, \( c^i_t = \{ c^i_t (h^t) \}_{t=0}^{\infty} \) for each \( h^t \in S^t \)
  - expected utility criterion:

\[
U (c^i) = \mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t u^i (c^i_t (h^t)) \right\} = \sum_{t=0}^{\infty} \sum_{h^t} \beta^t u^i (c^i_t (h^t)) \pi_t (h^t)
\]

where
- \( u' (c) > 0, u'' (c) < 0 \)
- \( \lim_{c \downarrow 0} u' (c) = +\infty \)

to ensure \( c_t > 0 \) for all \( t \)
Model Setup

A feasible allocation must satisfy

\[
\sum_{i=1}^{I} c_i^t (h^t) \leq \sum_{i=1}^{I} y_i^t (h^t)
\]

for all \( t \) and for all \( h^t \).
Remember?
A first-order necessary condition for Pareto optimum is

$$
\beta^t u' \left( c^i_t \left( h^t \right) \right) \pi_t \left( h^t \right) = \frac{\theta_t \left( h^t \right)}{\lambda_i}
$$

for all $i = 1, \ldots, I$, and for all $t \geq 0$ and all $h^t$. 
**Remember?**

Consider two agents, $i \neq j$. The ratio of their marginal utilities at each period, for all possible histories, is

$$\frac{u'(c^i_t(h^t))}{u'(c^j_t(h^t))} = \frac{\lambda_j}{\lambda_i}$$

This implies:

$$c^i_t(h^t) = u'^{-1} \left[ \frac{\lambda_j}{\lambda_i} u'(c^j_t(h^t)) \right]$$
Theorem

A Pareto optimal allocation is a function of the realized aggregate endowment and does not depend on

1. the particular history $h^t$ leading up to that outcome, nor
2. the realization of individual endowments,

so that if $h^t \neq h^\tau$ are such that $\sum_j y^j_t (h^t) = \sum_j y^j_\tau (h^\tau)$ then $c^i_t (h^t) = c^i_\tau (h^\tau)$. 
Corollary (First fundamental welfare theorem)

The competitive equilibrium is a particular Pareto optimal allocation, where $\mu_i = \lambda_i^{-1}$ for all $i = 1, \ldots, I$, is unique (up to a multiplication by a positive scalar). Furthermore, the shadow prices for the planner $\theta_t \left( h^t \right)$ are equal to Arrow-Debreu equilibrium prices $q^0_t \left( h^t \right)$.
What Next?

- We have already characterized Pareto-optimal allocation (PO).
- We have studied one assumption for a decentralized market economy: ADE
- Next we study alternative SME. Note along the way, connections btw SME and ADE via asset pricing relationships.
- W.t.s. FWT: Allocative “equivalence” between ADE and SME and PO.
Sequential Markets Economy

Market trading structure – assumptions:

- Trade occurs at each $t \in \mathbb{N}$.
- Trade in one-period complete Arrow securities.
- At each $t$ reached with history $h^t$, traders meet to trade for history $h^{t+1}$-contingent goods deliverable in $t + 1$. 
Example

\[ S = \{0, 1\} \]. At \( t = 2 \), trades occurs for only \( t = 3 \) goods at states that can be reached from the realized \( t = 2 \) history, \( h^2 = (0, 1, 1) \).
Preliminaries: Two new items ...

Now markets exist *sequentially*.

At start of each period $t$, after each $h^t$, traders need to keep track of what is *feasibly tradable*.

This depend on:

1. Wealth as a state variable.
2. A restriction that prevents forever-borrowing schemes by agents.

**Remark.** These two things not needed in ADE. Why?
Relevant state variable

- Need to find an appropriate *individual* state variable.
- This state variable tracks available opportunity set; to provide the right choices of consumption so that there will be enough resources left for future trades on contingent claims.
- State variable is the present value (in terms of history $h^t$ and date $t$) of expected current and future net claims – i.e. current wealth of the consumer.
ADE again! Agent $i$’s wealth at time-$t$ is just the time-$t$ expected value of all her current and future net claims conditional on time-$t$, history $h^t$:

$$\Omega^i_t (h^t) = \sum_{\tau=t}^{\infty} \sum_{h^\tau|h^t} q^t_\tau (h^\tau) d_\tau (h^\tau)$$

$$= \sum_{\tau=t}^{\infty} \sum_{h^\tau|h^t} q^t_\tau (h^\tau) [c^i_\tau (h^\tau) - y^i_\tau (h^\tau)]$$

But,

$$q^t_\tau (h^\tau) = \beta^{\tau-t} u' \left( c^i_\tau (h^\tau) \right) \pi_\tau \left( h^\tau | h^t \right)$$

Note $\Omega^i_t (h^t)$ has the same expression as the value of a tail asset!
Since, aggregate endowment must equal aggregate consumption (following any $h^t$), then

$$\sum_{i=1}^{I} \Omega^i_t (h^t) = 0.$$ 

for all $t$ and all $h^t$.

**Remarks:**

- The cross-sectional distribution of tail wealth across all agents $i$ sums to zero, since all contingent debt sellers are balanced out by buyers.
- When we move from this ADE to the SME, we can relate this tail wealth of each $i$, $\Omega^i_t (h^t)$, to the time $t$ history $h^t$ individual asset of the sequential markets world.
SBC and Debt limits

- In ADE, households face a single intertemporal budget constraint that ensures intertemporal solvency.
- In the sequential markets setting, there will be a sequence of budget constraints, indexed by $t$ and $h^t$.
- Need to ensure sequential asset trades are not open to “Ponzi schemes” – i.e. consumers cannot forever be consuming more than their endowments.
- We will consider the weakest possible restrictions called “natural debt limits”.

Definition (Natural debt limit)

Let the Arrow-Debreu price in terms of the time \( t \) history \( h^t \) numeraire good be \( q^t_\tau (h^\tau) \) for \( \tau \geq t \). The value of the tail of \( i \)'s endowment sequence at time \( t \) given history \( h^t \),

\[
A^i_t (h^t) = \sum_{\tau = t}^{\infty} \sum_{s^\tau | h^t} q^t_\tau (h^\tau) y^i_\tau (s^\tau)
\]

is the natural debt limit at time \( t \) and history \( h^t \).
Remarks

\[ A^i_t (h^t) = \sum_{\tau=t}^{\infty} \sum_{s^\tau|h^t} q^t_\tau (h^\tau) y^i_\tau (s^\tau) \]

- The maximal amount that \( i \) can repay his debt starting from time \( t \) is thus the tail value of his endowment starting out from time \( t \) given history \( h^t \).
- Alternatively, this says the worst \( i \) can do is to consume zero forever from time \( t \) to repay existing debt at time \( t \) history \( h^t \).
- At each time \( t \), \( i \) will face one such borrowing constraint for each possible realization \( h^{t+1} \) the next period.
Sequential trades: Timing and Actions

Markets are open for trade in one period-ahead state-contingent claims every period.

1. $h^t$ realized
2. Relevant claim $\tilde{a}_t^i(h^t)$ realized
3. Agent $i$ chooses $c_t^i(h^t)$ realized

$h^{t+1} = (h^t, s_{t+1})$

1. Endowment $y_t^i(h^t)$ realized
2. Relevant claim $\tilde{a}_t^i(h^t)$ realized
3. Agent $i$ chooses $\{\tilde{a}_{t+1}^i(s_{t+1}, h^t)\} \in \mathbb{R}^n$ realized

\[ \tilde{a}_{t+1}^i(h^{t+1}) \]
Suppose the pricing kernel $\tilde{Q}_t (s_{t+1} | h^t)$ exists.

$\tilde{Q}_t (s_{t+1} | h^t)$: price of one unit of time $t + 1$ consumption, contingent on realization of $s_{t+1}$ in $t + 1$, given $t$-history $h^t$.

Agent $i$’s sequence of budget constraints:

$$\tilde{c}_t^i (h^t) + \sum_{s_{t+1}} \tilde{a}_{t+1}^i (s_{t+1}, h^t) \tilde{Q}_t (s_{t+1} | h^t) \leq y_t^i (h^t) + \tilde{a}_t^i (h^t)$$

for $t \geq 0$, at each $h^t$. 
Note at time $t$, given $h^t$, $i$ chooses

- Current consumption: $\tilde{c}_i^t(h^t)$, and
- Quantities of all possible $n$ number of state-contingent claims next period:

$$\left(\tilde{a}_{i+1}^t(s_{t+1}, h^t)\right) \in \mathbb{R}^n$$
No-Ponzi borrowing constraint:

\[-\tilde{a}_{t+1}^i (s_{t+1}) \leq A_{t+1}^i (h^{t+1}) = \sum_{\tau=t+1}^{\infty} \sum_{h^\tau|h^{t+1}} q_{\tau}^{t+1} (h^\tau) y_{\tau}^i (h^\tau).\]

Huh? ...

- Amount of debt \(i\) brings into all possible \(s_{t+1} \in S\),
- \textit{Must} be repayable, in the worst case,
- by expected discounted (real) value of tail endowments, remaining from \(t + 1\) onward.
- Worst case: consume nothing forever from \(t + 1\) on ...! Merd!
Agent $i$ chooses $\{c_t^i(h^t), \tilde{a}_{t+1}^i(s_{t+1}, h^t)\}_{t=0}^{\infty}$ to:

$$\max \sum_{t=0}^{\infty} \sum_{h^t} \left\{ \beta^t u(\tilde{c}_t^i(h^t)) \pi_t(h^t) \right. \right.$$

$$+ \eta_t^i(h^t)[y_t^i(h^t) + \tilde{a}_t^i(h^t)$$

$$- \tilde{c}_t^i(h^t) - \sum_{s_{t+1}} \tilde{a}_{t+1}^i(s_{t+1}, h^t) \tilde{Q}_t(s_{t+1}|h^t) \right.$$ 

$$+ \nu_t^i(h^t; s_{t+1}) \tilde{a}_{t+1}^i(s_{t+1}, h^t) + A_{t+1}^i(h^{t+1}) \right\}$$

for given initial wealth $\tilde{a}_0^i(h^0)$. 
Remarks: Following each $h^t$,

- There are $n$ no-Ponzi constraints to consider. Why?
- So there are $n$ Lagrange multipliers $\nu^i_t (h^t; s_{t+1})$, one for each possible $s_{t+1} \in S$.
- For each $s_{t+1}$ need to calculate upper bound on negative assets:

$$A^i_{t+1} (h^{t+1}) = \sum_{\tau=t+1}^{\infty} \sum_{h^\tau|h^{t+1}} q^{t+1}_\tau (h^\tau) y^i_\tau (h^\tau).$$
Optimal decision by agents $i$:
No-Ponzi constraints not binding. Why? So $\nu^i_t (h^t; s_{t+1}) = 0$ for all $t$, all $h^t$.

Then necessary (and sufficient) condition for optimal consumption-asset-accumulation strategy is

$$\tilde{Q}_t (s_{t+1}|h^t) = \beta \frac{u' (\tilde{c}^i_{t+1} (h^{t+1}))}{u' (\tilde{c}^i_t (h^t))} \pi_t (h^{t+1}|h^t)$$

for all $s_{t+1}, t \geq 0$ and $h^t$.

Crickey! This is a familiar looking one-period pricing kernel we encountered in the Arrow-Debreu economy!
**Definition**

A **distribution of wealth** is a vector \( \tilde{\mathbf{a}}_t (h^t) = \{\tilde{a}_i^i (h^t)\}_{i=1}^I \) satisfying \( \sum_i \tilde{a}_i^i (h^t) = 0 \).

**Definition**

A **sequential trading competitive equilibrium** is an initial distribution of wealth \( \tilde{\mathbf{a}}_0 (s_0) \), an allocation (sequence of allocations for all agents) \( \{\tilde{c}_i^i\}_{i=1}^I \) and pricing kernels \( \tilde{Q}_t (s_{t+1} | h^t) \) such that

1. for all \( i \), given \( \tilde{\mathbf{a}}_0^i (s_0) \) and \( \tilde{Q}_t (s_{t+1} | h^t) \), the consumption allocation \( \tilde{c}_i^i = \{\tilde{c}_i^t\}_{t=0}^\infty \) solves agent \( i \)'s optimization problem;

2. for all realizations of \( \{h^t\}_{t=0}^\infty \) the agent’s consumption allocation and implied asset portfolios \( \{\tilde{c}_i^t (h^t), \{\tilde{a}_i^i (s_{t+1}, h^t)\}_{s_{t+1}}\}_{t \in \mathbb{N}} \) satisfy \( \sum_i \tilde{c}_i^i (h^t) = \sum_i y_i^i (h^t) \) and \( \sum_i \tilde{a}_i^i (s_{t+1}, h^t) = 0 \) for all \( s_{t+1} \).
**Theorem**

*The time-0 trading arrangement in the Arrow-Debreu equilibrium with complete markets has the same allocations as the sequential trading arrangement with one-period complete Arrow securities,*

\[
\{c^i\}_{i=1}^I = \{\widetilde{c}^i\}_{i=1}^I ,
\]

*for an appropriate initial distribution of wealth in the sequential markets equilibrium, \(\{\widetilde{a}_{0}(s_0)\}_{i=1}^I\).*
Proof.

First we show “ADE ⇒ SME”.

- Take Arrow-Debreu equilibrium \( q_t^0 (h^t) \) as given.

- Suppose \( \exists \tilde{Q}_t (s_{t+1}|h^t) \) satisfying recursion
  \[
  q_{t+1}^0 (h^{t+1}) = \tilde{Q}_t (s_{t+1}|h^t) q_t^0 (h^t)
  \]
  \[
  \iff \tilde{Q}_t (s_{t+1}|h^t) = \frac{q_{t+1}^0 (h^{t+1})}{q_t^0 (h^t)} = q_{t+1}^0 (h^{t+1}).
  \]

- To show guess is true, take Arrow-Debreu equilibrium first-order conditions from two successive periods and write:
  \[
  \beta \frac{u' (c_{t+1}^i (h^{t+1}))}{u' (c_t^i (h^t))} \pi_t (h^{t+1}|h^t) = \frac{q_{t+1}^0 (h^{t+1})}{q_t^0 (h^t)}
  \]
Proof (cont’d).

But then if guess is true, it must be that

\[ \beta \frac{u'(c_{t+1}^i(h^{t+1}))}{u'(c_t^i(h^t))} \pi_t(h^{t+1}|h^t) = \tilde{Q}_t(s_{t+1}|h^t) \]

\[ = \beta \frac{u'(\tilde{c}_{t+1}^i(h^{t+1}))}{u'(\tilde{c}_t^i(h^t))} \pi_t(h^{t+1}|h^t). \]

So then, Arrow-Debreu equilibrium is equivalent to the sequential markets equilibrium in terms of allocations,

\[ \left\{ c^i \right\}_{i=1}^I = \left\{ \tilde{c}^i \right\}_{i=1}^I. \]
Proof (cont’d).

Next we show “ADE ⇐ SME”:

- Pick \( \{ \widetilde{a}_0^i (s_0) \}_{i=1}^I \) s.t. SBCs in SME consistent with IBC for ADE. Guess that \( \{ \widetilde{a}_0^i (s_0) \}_{i=1}^I = 0_{I \times 1} \).

- Why? In Arrow-Debreu equilibrium, at time 0, agents bring in only their endowments, \( y_0^i (s_0) \).

- At \( t \geq 0 \) and history \( h^t \), \( i \) chooses asset portfolio, \( \widetilde{a}_{t+1}^i (s_{t+1}, h^t) = \Omega_{t+1}^i (h^{t+1}) \) for all \( s_{t+1} \).

- The expected value in date \( t \) terms is

\[
\sum_{s_{t+1}} \widetilde{a}_{t+1}^i (s_{t+1}, h^t) \quad \widetilde{Q}_t (s_{t+1} | h^t) = \sum_{s_{t+1}} \Omega_{t+1}^i (h^{t+1}) q_{t+1}^i (h^{t+1})
\]

\[
= \sum_{\tau=t+1}^{\infty} \sum_{h^\tau | h^t} q_{\tau}^t (h^\tau) \left[ c_{\tau}^i (h^\tau) - y_{\tau}^i (h^\tau) \right]
\]

just the tail value of wealth!
Proof (cont’d).

• Show that \( i \) can afford this portfolio strategy. Use SME SBCs. At time 0 given \( \tilde{a}^i_0 (s_0) = 0 \),

\[
\tilde{c}^i_0 (s_0) + \sum_{t=1}^{\infty} \sum_{h^t} q^0_t (s_t) \left[ c^i_t (h^t) - y^i_t (h^t) \right] = y^i_t (s_0) + 0
\]

• But this is the same as IBC in the ADE.

• So \( \tilde{c}^i_0 (s_0) = c^i_0 (s_0) \).
Proof.

- For all $t > 0$, we can write $\tilde{a}^i_t (h^t) = \Omega^i_t (h^t)$, and the time $t$, $h^t$-BC is

$$\tilde{c}^i_t (h^t) + \sum_{s_{t+1}} \tilde{a}^i_{t+1} (s_{t+1}, h^t) \tilde{Q}_t (s_{t+1}|h^t) = y^i_t (h^t) + \tilde{a}^i_t (h^t)$$

$$\Rightarrow \sum_{s_{t+1}} \tilde{a}^i_{t+1} (s_{t+1}, h^t) \tilde{Q}_t (s_{t+1}|h^t) = \Omega^i_t (h^t) - [\tilde{c}^i_t (h^t) - y^i_t (h^t)]$$

$$\Rightarrow \sum_{s_{t+1}} \Omega^i_{t+1} (h^{t+1}) q^i_{t+1} (h^{t+1}) = \Omega^i_t (h^t) - [\tilde{c}^i_t (h^t) - y^i_t (h^t)]$$

$$\Rightarrow \sum_{\tau=t+1}^{\infty} \sum_{h^\tau|h^t} q^i_\tau (h^\tau) \left[ c^i_\tau (h^\tau) - y^i_\tau (h^\tau) \right]$$

$$= \Omega^i_t (h^t) - [\tilde{c}^i_t (h^t) - y^i_t (h^t)]$$

It then follows that $\tilde{c}^i_t (h^t) = c^i_t (h^t)$ for all $t$ and $h^t$. 

\[\square\]
Notes

The equivalence between Arrow-Debreu equilibrium and Arrow's sequential markets equilibrium follows from two key factors:

- Agents are v.N-M expected utility maximizers – their once-and-for-all time 0 choices are *time consistent*. Past actions affect future payoffs but future actions do not affect past payoffs.

- Under complete markets, the budget sets defined by the two formulations are equivalent, and thus Arrow-Debreu equilibrium prices of contingent claims are equal to Arrow’s spot prices weighted by the price in period 1 of the appropriate Arrow security.
Recursive (Markov) Competitive Equilibrium

- The assumptions about the state variables so far in the Arrow-Debreu equilibrium and sequential markets equilibrium economies are too general to be useful – at each $t$, state variables are made up of the entire history leading up to $t$, i.e. $h^t := (s_0, s_1, ..., s_t)$.

- For practical purposes, we need to discipline the evolution of the state further – e.g. to be Markovian – so only a few state variables suffice to describe the position of the economy at each time period.

- We’ll look at a recursive competitive equilibrium formulation of the sequential markets equilibrium and Arrow-Debreu equilibrium.
Endowments with Markov property

Consider the state space, \( S \). So exogenous event is \( s \in S \). Let \( s \) be governed by a Markov chain:

- \( \pi_0 (s) = \Pr (s_0 = s) \).
- \( \pi (s'|s) = \Pr (s_{t+1} = h'|s_t = s) \).

The Markov chain induces a sequence of probability measures on histories \( h^t \). The probability of realizing history \( h^t \) is

\[
\pi_t (h^t) = \pi (s_t|s_{t-1}) \pi (s_{t-1}|s_{t-2}) \cdots \pi (s_1|s_0) \pi_0 (s_0).
\]

where it is assumed \( \pi_0 (s_0) = 1 \).
The Markov property says that

$$\pi_t (h^t | h^k) = \pi (s_t | s_{t-1}) \pi (s_{t-1} | s_{t-2}) \cdots \pi (s_{k+1} | s_k)$$

where $\pi_t (h^t | h^k)$ depends only on state $s_k$ at $k < t$ and the history prior to $k$ is redundant.

**Example**

$$\pi_3 (h^3 | h^2) = \pi_3 ((s_0, s_1, s_2, s_3) | (s_0, s_1, s_2)) = \pi (s_3 | s_2).$$

Then, for each $i = 1, \ldots, I$, we can write endowments as

$$y^i_t (h^t) = y^i (s_t)$$

Since $s_t$ is a Markov process, $y^i_t (s_t)$ will also be a Markov process.
Equilibrium inherits Markov property

**Theorem**

Given $y^i_t (s^t)$ a Markov process, the Arrow-Debreu equilibrium price of date-$\tau$ history $h^\tau$ consumption goods in terms of date $t$, $0 \leq t \leq \tau$, history $h^t$ goods is not history dependent:

$q^t_\tau (h^\tau) = q^k_j \left( \tilde{h}^k \right)$ for $j, k \geq 0$ such that $\tau - t = k - j$ and $(s_t, s_{t+1}, ..., s_\tau) = (\tilde{s}_j, \tilde{s}_{j+1}, ..., \tilde{s}_k)$. 
Remark. Natural debt limits and household wealth are also history independent: \( A^i_t(s^t) = A^i(s^t) \) and \( \Omega^i_t(s^t) = \Omega^i(s^t) \).

- Each agent enters every period with wealth independent of past endowment realizations.
- Past trades have fully insured away all idiosyncratic endowment risks.
- So an agent enters the current period with current-state contingent wealth just sufficient to fund a trading scheme that insures against future idiosyncratic risks.
- The pricing kernel \( Q(s_t | s_{t-1}) \) thus provides the correct signal, along with market clearing, to coordinate trade in time \( t - 1 \) such that all idiosyncratic risks are eliminated.
- However, if there are aggregate risks, they would still have to be borne by all agents.
Now $y^t_i(s_t)$ Markov.

Agent $i$’s competitive equilibrium sequence problem, given $Q(s'|s)$, is now recursive:

$$v^i(a, s) = \max_{c, \hat{a}(s')} \left\{ u(c) + \beta \sum_{s'} v^i(\hat{a}(s'), s') \pi(s'|s) \right\}$$

subject to

$$c + \sum_{s'} \hat{a}(s') Q(s'|s) \leq y^i(s) + a$$

$$c \geq 0$$

$$-\hat{a}(s') \leq A^i(s') \text{ for all } s' \in S.$$
Let the optimal decision rules associated with the fixed-point solution of the Bellman equation be

\[ c = h^i (a, s) \]

\[ \hat{a} (s') = g^i (a, s, s') \]

for each \( i = \{1, \ldots, I\} \).
We can show that this optimal solution depends on the price kernel $Q(s' | s)$.

Evaluate the first-order condition for the RHS of each Bellman equation and apply the Benveniste-Scheinkman formula to get

$$Q (s_{t+1} | s_t) = \beta \frac{u' (c^{i}_{t+1})}{u' (c^{i}_{t})} \pi (s_{t+1} | s_t)$$

where $c = h^{i} (a, s)$ and $\tilde{a} (s') = g^{i} (a, s, s')$. 
Definition

A recursive competitive equilibrium is an initial distribution of wealth $\{a_0^i\}_{i=1}^I$, a pricing kernel $Q(s'|s)$, sets of value functions $\{v^i(a, s)\}_{i=1}^I$, and decision rules $\{h^i(a, s), g^i(a, s, s')\}_{i=1}^I$ such that

1. for all $i$, given $a_0^i$ and the pricing kernel, the decision rules solve the household $i$'s problem;

2. for all histories $\{s_t\}_{t=0}^\infty$, the consumption and assets $\{\{c_t^i, \hat{a}_{t+1}^i(s')\}_{s'}\}_{t=0}^\infty$ implied by the decision rules satisfy $\sum_i c_t^i = \sum_i y^i(s_t)$ and $\sum_i \hat{a}_{t+1}^i(s') = 0$ for all $t$ and $s'$. 
**$j$-step ahead pricing kernel**

- Since a complete set of markets exists for all $j$ periods ahead contingent claims,
- a consumer $i$, at the end of period $t$, can always buy $z_{t,j}^i(s_{t+j})$ units of contingent consumption claims, for $j \geq 1$.
- Recall that agent $i$’s sequential budget constraint is

\[
c_t^i + \sum_{s_{t+1}} Q_1(s_{t+1}|s_t)a_{t+1}^i(s_{t+1}) \leq y^i(s_t) + a_t^i.
\]
The agent’s next period wealth depends on next period state \( s_{t+1} \) and the composition of the asset portfolio:

\[
a^i_{t+1}(s_{t+1}) = z^i_{t,1}(s_{t+1}) + \sum_{j=2}^{\infty} \sum_{s_{t+j}} Q_{j-1}(s_{t+j}|s_{t+1}) z^i_{t,j}(s_{t+j}).
\]

So the outcome \( s_{t+1} \) will determine which element of the \( n \)-dimensional vector

\[
\begin{bmatrix}
z^i_{t,1}(s_{t+1} = s_1) \\
\vdots \\
z^i_{t,1}(s_{t+1} = s_n)
\end{bmatrix}
\]

pays off at time \( t + 1 \).

But this is only one component of time \( t + 1 \) wealth.
\[
a_t(s_{t+1}) = z_t(s_{t+1}) + \sum_{j=2}^{\infty} \sum_{s_{t+j}} Q_{j-1}(s_{t+j} | s_{t+1}) z_{t,j}(s_{t+j}).
\]

- The second term on the RHS, conditional of outcome \( s_{t+1} \) when time \( t + 1 \) arrives, is the expected (capital gains or losses) from holding longer term claims on \( t + 1 + j, j \geq 1 \), consumption.
- Together they make up next-period – i.e. \( t + 1 \) – wealth if \( s_{t+1} \in S \) is to be realized.
- Using this fact, the sequence of budget constraints becomes

\[
c_t^i + \sum_{j=1}^{\infty} \sum_{s_{t+j}} Q_j(s_{t+j} | s_t) z_{t,j}(s_{t+j}) \leq y^i(s_t) + a_t^i.
\]
Note that the first-order condition for an optimal plan by agent $i$ will imply that

$$Q_j(s_{t+j}|s_t) = \beta \sum_{s_{t+1} \in S} \frac{u'[c_t^{i}(s_{t+1})]}{u'[c_t^{i}(s_t)]} \pi(s_{t+1}|s_t)Q_{j-1}(s_{t+j}|s_{t+1}).$$

Since agent’s optimal RCE strategy also requires satisfying,

$$Q_1(s_{t+1}|s_t) := Q(s_{t+1}|s_t) = \beta \frac{u'(c_{t+1}^{i})}{u'(c_t^{i})} \pi(s_{t+1}|s_t)$$

we then have

$$Q_j(s_{t+j}|s_t) = \sum_{s_{t+1} \in S} Q_1(s_{t+1}|s_t)Q_{j-1}(s_{t+j}|s_{t+1}),$$

a recursive formula for computing $j$-step ahead pricing kernels for $j = 2, 3, ....$
Arbitrage-free pricing and redundant assets

- Suppose, apart from purchasing $z_{t,j}(s_{t+j})$ units of $j$-step ahead complete Arrow securities, Sam also trades an ex-dividend stock called a Lucas tree.
- A unit of this stock allows Sam to have the right to a unit of fruit or dividend $d(s_{t+1})$ from this Lucas tree, if state $s_{t+1}$ occurs.
- Sam can buy $N_t$ units of this stock. The ex-dividend price is $p(s_t)$.
- So Sam can obtain $N_t[p(s_{t+1}) + d(s_{t+1})]$ units of consumption in $t + 1$ if $s_{t+1}$ occurs then.
See notes and LS, Ch.8 for details....

In equilibrium, two arbitrage-free pricing conditions

\[
p(s_t) = \sum_{s_{t+1}} Q_1(s_{t+1} | s_t) [p(s_{t+1}) + d(s_{t+1})],
\]

\[
Q_j(s_{t+j} | s_t) = \sum_{s_{t+1}} Q_{j-1}(s_{t+j} | s_{t+1}) Q_1(s_{t+1} | s_t), \quad j = 2, 3, \ldots
\]

for all \( t \in \mathbb{N} \).

Meaning?